Homework II
Due Date: 22/03/2024

Exercise 1. Find the Green's functions to the Poisson equation with Dirichelt boundary condition in each of the following domains $\Omega$ .
(i) (1 point) $\Omega = \{(x,y,z) : x > 0, y > 0, z > 0\}$.
(ii) (1 point) $\Omega = \{(x,y,z) : x^2 + y^2 + z^2 < R^2, x > 0, y > 0, z > 0\}$.

Exercise 2. Consider the following problems.
(i) (1 point) Solve

$$\begin{cases} \partial_t u - \partial_x^2 u = 3t^2, & t > 0, x \in \mathbb{R}, \\ u(0,x) = \sin x, & x \in \mathbb{R}. \end{cases}$$

(ii) (1 point) Solve

$$\begin{cases} \partial_t u - \partial_x^2 u = 0, & t > 0, x > 0, \\ u(0,x) = 0, & x > 0. \\ u(t,0) = 1, & t > 0. \end{cases}$$

Exercise 3. We say $v \in C^2(\overline{\Omega})$ is subharmonic if

$$-\Delta v \leq 0, \quad x \in \Omega.$$

(i) (1 point) Prove for a subharmonic $v$ that

$$v(x) \leq \frac{1}{|B_r|} \int_{B_r(x)} v(y)dy,$$

for all $B_r(x) \subset U$.
(ii) (1 point) Prove that

$$\max_{\overline{U}} v = \max_{\partial U} v.$$

(iii) (1 point) Assume $\Omega = \{(x,y) : x^2 + y^2 < 1\}$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a classical solution of the Dirichlet boundary problem

$$-\Delta u = -x^2, \quad x \in \Omega,$$
$$u = x^2 + y^2 - 5xy + 2, \quad x \in \partial\Omega.$$

Calculate the maximal value of $u$ in $\Omega$.

Exercise 4. We say $v \in C^{1,2}((0,T) \times \Omega)$ is a subsolution of the heat equation if

$$\partial_t v - \Delta v \leq 0, \quad x \in (0,T) \times \Omega.$$

(i) (1 point) Prove for a subsolution $v$ that

$$v(t,x) \leq \frac{1}{4r^n} \iint_{E_r(t,x)} v(s,y) \frac{|x-y|^2}{(t-s)^2} dyds,$$

for all $E_r(t,x) := \{(s,y) \in \mathbb{R}_+ \times \mathbb{R}^n : s \leq t, \Gamma(t-s, x-y) \geq \frac{1}{r^n}\} \subset (0,T] \times \Omega$, where $\Gamma(t-s, x-y)$ is the fundamental solution of the heat equation.
(ii) (1 point) Prove that

$$\max_{[0,T]\times\overline{\Omega}} v = \max_{[0,T]\times\overline{\Omega}/(0,T]\times\Omega} v.$$

(iii) (1 point) Assume $u \in C^{1,2}((0,\infty)\times(0,1))$ is a classical solution of the Dirichlet boundary problem

$$\begin{aligned} \partial_t u - \partial_x^2 u = -x^2, \quad t > 0, 0 < x < 1, \\ u(0,x) = \sin \pi x, \quad 0 < x < 1, \\ u(t,0) = 2te^{1-t}, u(t,1) = 1 - \cos \pi t, \quad t > 0. \end{aligned}$$

Calculate the maximal value of $u$ in $[0,1]\times[0,1]$.